Recap Symmetric Spaces

Tuesday, 26 May 2020 10:09

RSS RSP OSL

Definition 2.8. Let M be a connected Riemannian manifold. Then M is *locally symmetric* if for all $p \in M$ there is a normal neighbourhood U of $p \in M$ and an isometry $s_p : U \to U$ such that $s_p^2 = \mathrm{id}$ and such that p is the unique fixed point of s_p in U. The space M is *globally symmetric* if it is locally symmetric and each s_p can be extended to an isometry of M.

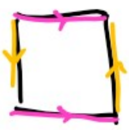
RSS

Ex: $\mathbb{S}^2$,

Note: There may be more than one global fixed point of s_p.

2-dim RSS: • $\mathbb{E}^2$, $\mathbb{S}^2$, $\mathbb{H}^2$, $\underbrace{\mathbb{S}^1 \times \mathbb{S}^1,\ \mathbb{R} \times \mathbb{S}^1,\ \text{Klein bottle}}_{\text{euclidean type}}$, $\mathbb{RP}^2$ (compact type.)

$\mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ $\qquad \Gamma \curvearrowright \mathbb{R}^2$

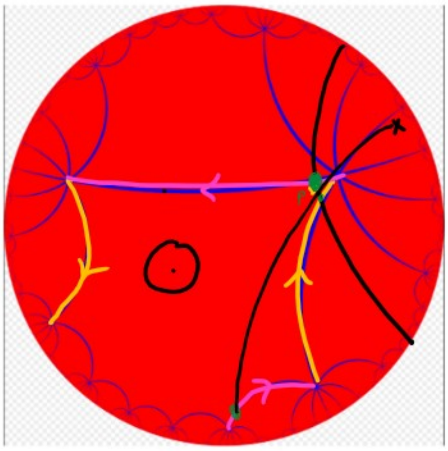


$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{\tilde{s}_p} & \mathbb{R}^2 \\ \pi \downarrow & \searrow f & \downarrow \pi \\ \mathbb{R}^2/\Gamma & \xrightarrow{s_p} & \mathbb{R}^2/\Gamma \end{array}$$

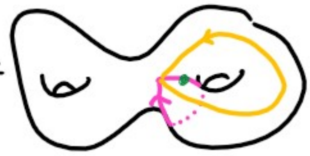
$f = \tilde{s}_p \circ \pi$ $\qquad \forall x \in \mathbb{R}^2$

If $\forall \gamma \in \Gamma$: $f(\gamma(x)) = f(x)$

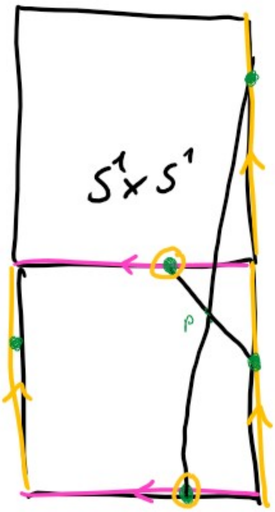
then s_p is well defined

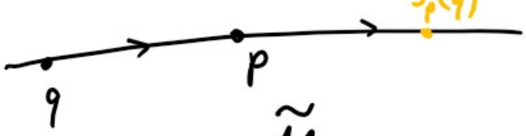


$M = \mathbb{H}^2/\Gamma$, =



are locally symmetric.





M locally symmetric space, $\tilde{M}/\Gamma = M$,

$\tilde{M}$ is a RSS.

$\mathbb{RP}^2 = \mathbb{S}^2/\pm 1$

Definition 2.31. Let G be a connected Lie group and let $K \leq G$ be a closed subgroup of G. Then (G, K) is a *Riemannian symmetric pair* if

(i) $\mathrm{Ad}_G(K) \leq \mathrm{GL}(\mathfrak{g})$ is compact, and
(ii) there exists an involution $\sigma : G \to G$ with $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$.

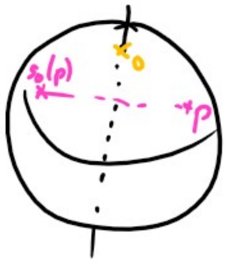
RSP

Note: If K is compact, then $\mathrm{Ad}(K)$.

$(\widetilde{SL(2,\mathbb{R})}, \underbrace{\widetilde{SO(2)}}_{\mathbb{R}})$ is RSP

Ex: $\mathbb{S}^n$: $o = e_{n+1}$ $\quad n \geq 2$ $\qquad$ RSS $\longrightarrow$ RSP

Ex: $\mathbb{S}^n$: $\overset{n\geq 2}{o = e_{n+1}}$ RSS $\longrightarrow$ RSP $\mathbb{K}$



$s_o : \mathbb{S}^n \longrightarrow \mathbb{S}^n$

$p \longmapsto \begin{pmatrix} -Id_n & \\ & 1 \end{pmatrix} \cdot p$

$SO(n) \cong SO(n) \times \{1\}$ "

$$G := Is(\mathbb{S}^n)^0 = O(n+1)^0 = SO(n+1)$$

$$K := Stab_G(o) = \{ g \in SO(n+1) : g e_{n+1} = e_{n+1} \} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in SO(n) \right\}$$

$$g = \begin{array}{cc} n & 1 \\ \multicolumn{2}{c}{\left(\begin{array}{c|c} A & B \\ \hline C & d \end{array}\right)} \end{array}, \quad g e_{n+1} = e_{n+1} \iff \begin{pmatrix} B \\ d \end{pmatrix} = g e_{n+1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} d = 1 \\ B = 0. \end{matrix}$$

$$g^T g = Id : \begin{pmatrix} A^T & C^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ C & 1 \end{pmatrix} = \begin{pmatrix} A^T A + C^T C & C^T \\ C & 1 \end{pmatrix} = Id \Rightarrow (G, K) \text{ is a RSP.}$$

$$\sigma : G \longrightarrow G, \quad G^\sigma = \{ g \in SO(n+1) : \sigma(g) = g \} = \left\{ \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} \in SO(n+1) \right\}$$

$$g \longmapsto s_o g s_o$$

$$\sigma(g) = \begin{pmatrix} -Id_n & \\ & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & d \end{pmatrix} \begin{pmatrix} -Id_n & \\ & 1 \end{pmatrix} = \begin{pmatrix} -Id_n & \\ & 1 \end{pmatrix} \begin{pmatrix} -A & B \\ -C & d \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & d \end{pmatrix}$$

$$G^\sigma = \left\{ \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix} : d = \pm 1, A \in O(n), \det(A) \cdot d = 1 \right\} \text{ is not connected.}$$

$$(G^\sigma)^0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in SO(n) \right\} = K \underset{\neq}{<} G^\sigma.$$

Ex: $M = \mathbb{R}P^2 = \mathbb{S}^2 / \pm 1$. $o = \pm e_3$, $s_o : \mathbb{R}P^2 \longrightarrow \mathbb{R}P^2$

$$[p] \longmapsto \left[\begin{pmatrix} -Id_n & \\ & 1 \end{pmatrix} \cdot p \right] = \left[\begin{pmatrix} Id_n & \\ & -1 \end{pmatrix} \cdot p \right]$$

$$G = Is(M)^0 = \left(O(3) / \pm Id \right)^0 = O(3) / \pm Id$$

$$K = Stab(o) = \{ g \in O(3)/\pm Id : g . o = o \} = \left\{ \left[\begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix} \right] \in O(3)/\pm Id \right\},$$

You cannot connect $\left[\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right]$ and $\left[\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \right] = \left[\begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right]$

$$G^\sigma = \{ g \in O(3)/\pm Id : s_o \cdot g \cdot s_o = g \} = K$$

$$\Rightarrow (G^\sigma)^0 \underset{\neq}{\leq} K = G^\sigma.$$

Ex: $(G^\sigma)^0 \underset{\neq}{\leq} K \underset{\neq}{\leq} G^\sigma$ for $M = \mathbb{S}^2 \times \mathbb{R}P^2$

RSP $\longrightarrow$ RSS

$G/K =: M$ and Riemannian metric is determined by a

$G/K =: M$ and Riemannian metric is determined by a $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{p} \hat{=} T_o M$.

Definition 2.53. Let $\mathfrak{g}$ be a Lie algebra. A subalgebra $\mathfrak{k} \leq \mathfrak{g}$ is *compactly embedded* if $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ is the Lie algebra of a compact subgroup of $\mathrm{GL}(\mathfrak{g})$.

OSL

Definition 2.54. An *orthogonal symmetric Lie algebra* is a pair $(\mathfrak{g}, \Theta)$ consisting of a real Lie algebra $\mathfrak{g}$ and an involutive automorphism $\Theta \neq \mathrm{Id}$ of $\mathfrak{g}$ such that $\mathfrak{u} = \mathrm{E}_1(\Theta)$ $= \mathfrak{k}$ is compactly embedded in $\mathfrak{g}$. An orthogonal symmetric Lie algebra is *effective* if $Z(\mathfrak{g}) \cap \mathfrak{u} = 0$.

Def: $(\mathfrak{g}, \Theta)$ is reduced if there is no non-zero ideal of $\mathfrak{g}$ in $\mathfrak{k}$.

RSP $\longrightarrow$ OSL

$\mathfrak{g} = \mathrm{Lie}(G)$, $\Theta = D_e\sigma$.

OSL $\longrightarrow$ RSP

$(\mathfrak{g}, \Theta)$ OSL. (effective), semi-simple.. | Can choose

$G := \mathrm{Aut}(\mathfrak{g})^0$, $\sigma : G \longrightarrow G$, $g \longmapsto \Theta g \Theta$, $(G^\sigma)^0 < K < G^\sigma$.

If $\mathfrak{g}$ is semi-simple: Then $\mathrm{Aut}(\mathfrak{g})^0 \hat{=} \mathrm{Ad}(G) \Rightarrow \mathfrak{g} \hat{=} \mathrm{Lie}(G)$

effective: $Z(G) = 0$ $\quad$ "$G/Z(G)$"

$\Rightarrow G \hat{=} \mathrm{Aut}(\mathfrak{g})^0$.

$\mathrm{Lie}(\tilde{G}) = \mathfrak{g}$, then $\mathrm{Aut}(\mathfrak{g})^0 \hat{=} \mathrm{Ad}(\tilde{G}) \Rightarrow \mathrm{Lie}(G) = \mathrm{Lie}(\tilde{G})$

"Injectivity" for RSS $\longrightarrow$ OSL

Let M_1, M_2 be two RSS of non-compact type.

$\Rightarrow$ OSL's $(\mathfrak{g}_1, \Theta_1)$, $(\mathfrak{g}_2, \Theta_2)$.

If $(\mathfrak{g}_1, \Theta_1) \hat{=} (\mathfrak{g}_2, \Theta_2)$ isomorphic, M_1 and M_2 have the metric comming from the $B_{\mathfrak{g}}$.

$\Longrightarrow M_1$ is Riemannian isometric to M_2.

Definition 2.57. Let $(\mathfrak{g}, \Theta)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ and Killing form $B_{\mathfrak{g}}$.

(i) If $B_{\mathfrak{g}} \ll 0$ then $(\mathfrak{g}, \Theta)$ is of *compact type*. $\}$ semi-simple

Definition 2.57. Let $(\mathfrak{g}, \Theta)$ be an effective orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ and Killing form $B_{\mathfrak{g}}$.

(i) If $B_{\mathfrak{g}} \ll 0$ then $(\mathfrak{g}, \Theta)$ is of *compact type.*
(ii) If $B_{\mathfrak{g}}$ is non-degenerate and $B_{\mathfrak{g}}|_{\mathfrak{e} \times \mathfrak{e}} \gg 0$ then $(\mathfrak{g}, \Theta)$ is of non-compact type.
(iii) If $\mathfrak{e}$ is an abelian ideal in $\mathfrak{g}$ then $(\mathfrak{g}, \Theta)$ is of Euclidean type.

} semi-simple

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{u} \oplus \mathfrak{e} = E_1(\theta) \oplus E_{-1}(\theta).$$

If effective $\Rightarrow$ $B_{\mathfrak{g}}|_{\mathfrak{k} \times \mathfrak{k}} \ll 0$.

Theorem 2.60. Let $(\mathfrak{g}, \Theta)$ be an effective orthogonal symmetric Lie algebra. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is a direct sum of Θ-stable ideals such that:

(i) The $\mathfrak{g}_\mu$ $(\mu \in \{0, +, -\})$ are mutually orthogonal with respect to $B_{\mathfrak{g}}$.
(ii) The pairs $(\mathfrak{g}_0, \Theta|_{\mathfrak{g}_0})$, $(\mathfrak{g}_+, \Theta|_{\mathfrak{g}_+})$, $(\mathfrak{g}_-, \Theta|_{\mathfrak{g}_-})$ are orthogonal symmetric Lie algebras of Euclidean, non-compact and compact type resepectively.

Choose a Ad(K)-invariant scalar product $\langle\, , \rangle$ on $\mathfrak{p}$.

$B_{\mathfrak{g}}(X, Y) = \langle AX, Y \rangle$, A is symmetric.

$\Rightarrow$ diagonalize A, look at sign of eigenvalues of A,

$\Rightarrow$ the corresponding eigenspaces $\Rightarrow$ $\mathfrak{p}_0, \mathfrak{p}_-, \mathfrak{p}_+$.

...

Theorem 2.73. Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{g}$ is semisimple and $\mathfrak{k}$ does not contain an ideal of $\mathfrak{g}$. Then there are ideals $(\mathfrak{g}_i)_{i \in I}$ of $\mathfrak{g}$ such that

reduced.

(i) $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$,
(ii) the $(\mathfrak{g}_i)_{i \in I}$ are pairwise orthogonal with respect to $B_{\mathfrak{g}}$ and θ-invariant, and
(iii) $(\mathfrak{g}_i, \theta|_{\mathfrak{g}_i})$ is an irreducible orthogonal symmetric Lie algebra.

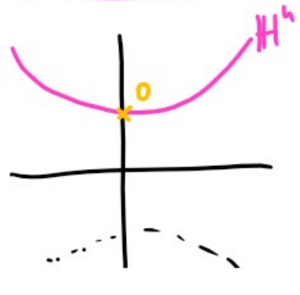
Moreover this decomposition is unique.

Globalize decomposition.

M RSS $\Rightarrow$ OSL $(\mathfrak{g}, \theta)$, reduced.

If M is simply connected $\Rightarrow$ $M = \underset{\substack{\parallel \\ \mathbb{R}^n}}{M_0} \times M_+ \times M_- = \mathbb{R}^n \times \sum_i M_i$

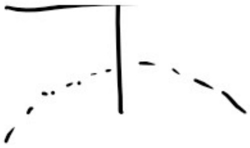
Duality:



Let b be a bilinear form on $\mathbb{R}^{n+1}$. $b(x,y) = x^T \begin{pmatrix} -I_n & \\ & 1 \end{pmatrix} y$

$$\mathbb{H}^n = \{ x \in \mathbb{R}^{n+1} : b(x,x) = -1,\ x_{n+1} > 0 \}$$

$$Is(\mathbb{H}^n) = \{ g \in M_{n+1} : b(gx, gy) = b(x,y)\ \forall x, y \in \mathbb{H}^n \}$$



$o = e_{n+1}$

$$\mathrm{Is}(\mathbb{H}^n) = \{ g \in M_{n+1} : b(gx, gy) = b(x,y) \ \forall x,y \in \mathbb{H}^n \}$$
$$= \left\{ g : \underline{g^T \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix} g = \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix}} \right\}$$
$$=: O(n,1)$$

$$s_0 : \mathbb{H}^n \to \mathbb{H}^n, \quad p \mapsto \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix} \cdot p, \qquad G = \mathrm{Is}(\mathbb{H}^n)^0 = O(n,1)^0 = O(n,1).$$

$$\mathfrak{g}_0 = \mathrm{Lie}(G) = \mathfrak{o}(n,1) = \left\{ X \in M_{n+1} : \underline{X^T \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix} + \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix} X = 0} \right\}$$

$$\Theta = D_e \sigma, \qquad \sigma(g) = s_0 g s_0 = \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix} g \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix}$$

$$\Theta(X) = \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix} X \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix}. \qquad \Theta\begin{pmatrix} A & B \\ C & d \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & d \end{pmatrix} = \begin{pmatrix} A & B \\ C & d \end{pmatrix}$$

$$\mathfrak{k} = E_1(\theta) = \{ X \in \mathfrak{o}(n,1) : \Theta(X) = X \} = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} : A^T + A = 0 \right\}$$

$$\mathfrak{p} = E_{-1}(\theta) = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \in \mathfrak{o}(n,1) : B^T = C \right\} = \left\{ \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} ; B \in \mathbb{R}^{n \times 1} \right\}$$

$$0 = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^T \begin{pmatrix} -\mathrm{Id} & \\ & 1 \end{pmatrix} + \begin{pmatrix} -\mathrm{Id}_n & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} & C^T \\ -B^T & \end{pmatrix} + \begin{pmatrix} 0 & -B \\ C & 0 \end{pmatrix},$$

$$\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{p}, \quad \text{Complexification} \quad \mathfrak{g} = \mathfrak{g}_0 + i \cdot \mathfrak{g}_0$$
$$= \mathfrak{k} \oplus \mathfrak{p} \oplus i \cdot \mathfrak{k} \oplus i \cdot \mathfrak{p}.$$

$$\mathfrak{g} = \mathfrak{o}(n,1) + i \cdot \mathfrak{o}(n,1).$$

$$\mathfrak{g}^* := \mathfrak{k} \oplus i \cdot \mathfrak{p} = \left\{ \begin{pmatrix} A & iB \\ iB^T & 0 \end{pmatrix} : A^T + A = 0 \right\}$$

Thm $\Rightarrow (\mathfrak{g}_0, \Theta)^* = (\mathfrak{g}^*, \text{complex conj})$ is again an OSL.

to be continued ...